Proof and proving in Algebra

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Introduction

The theme of proof has been at the core of a number of studies, different aspects have been discussed and different perspectives have been proposed. Although not always researchers found an agreement, certainly some basic ideas can be considered as shared and among others the assumption that proving constitutes an important part of mathematical activity, so that education to mathematical thinking, even if not reduced to formal proving, cannot ignore it. This shared assumption should have and in some cases had (Hoyles, 1997) important consequences on school practice, both in terms of curricula and in terms of classroom organization.

Still the didactical problem seems far to be solved. Teachers have troubles introducing pupils to proof and, generally speaking, to a theoretical perspective.

The analogy, but also possible discrepancies between a "natural" approach and mathematical approach to argumentation has been widely discussed. Clear evidence has been provided of the difficulties related to the distance between the way of spontaneously supporting a statement and the specific way mathematicians have elaborated in order to make a statement acceptable within a specific theoretical framework (Hanna, 1989; Duval, 1992; Balacheff, 1987; 1999; Harel & Sowder, 1998, Mariotti, 2001b).

In this perspective, the paper present and discuss on the pupils` introduction to proof and it will do it considering an unusual math context for proving, Algebra, and the specific support of a microworld. This contribution is based on a long-term study, carried out in collaboration with Michele Cerulli, who in particular developed a prototype microworld, L'Algebrista (Cerulli, 2004). The aim of this paper is that of discuss on some aspects related to the design of the specific technological environment; in particular, I'd like to discuss how the design of the microworld has been influenced by the basic assumption about its functioning as a tool of semiotic mediation for *proving in Algebra*, and generally speaking for the idea of *Theory*.

The notion of semiotic mediation, introduced by Vygotsky (1978), has been elaborated in the last years in the field of mathematics education, in particular let me refer to a previous discussion, that I published not long ago (Mariotti, 2002), where the process of semiotic mediation is explained referring to the use of computational tools, and microworlds in particular.

The key element on which the process of semiotic mediation is based concerns on the one hand, the link between tools and meanings emerging from their use in classroom activities and on the other hand the mathematical notions, which are the objective of instruction. That means that a research study, concerning the use of tools in the perspective of semiotic mediation, requires a complex analysis that can be summarized in the following components:

1. Epistemological analysis. That will refer to both the mathematical notions we intend to deal with and in the specific artefact [1] we intend to use in the classroom.
2. Cognitive analysis. The use of an artefact according a specific purpose generates schemes of utilization, contributing to the emergence of meanings, but also create a meaningful environment within which interpersonal discourse may evolve.
3. Didactical analysis. The introduction and integration of an artefact in classroom activities has to be carefully planned, so that it can contribute to develop the intended mathematical knowledge.

In the following, articulating these directions of analysis I will illustrate some key points concerning the design of a microworld.

Firstly, I will explain in what sense I will speak of Algebra as a Theory and then I will illustrate how a microworld might provide tools of semiotic mediation related to the key ideas of acting within an Algebra Theory and more generally in a theoretical perspective. Some examples, drawn from a long term teaching experiments will support our discussion.

Algebra as a theory

Since antiquity, Geometry has been considered a prototype of theoretical systematization of mathematical knowledge, the archetype of what in modern terms is called an Axiomatic. On the contrary, Algebra found its systematization relatively late in the history, but mainly there is no tradition of a theoretical approach to Algebra at the pre-university level, where the study of Algebra is often relegated to its operative aspects of “symbolic calculation”. This is what happens in Italian schools, where a lot of time is devoted to train pupils to get an expertise in rote calculations, most of the time empty of any sense. Negative consequences of this approach are well known and have been highlighted in different studies.

Taking an *operational-structural* perspective (Sfard, 1991), one can say that the operational character of pupils' conceptions related to algebraic formula and expressions tends to persist (Sfard & Linchevski, 1994, p.199), while, although symbolic manipulations of algebraic expressions is largely present in school practice, the absence of "structural conceptions" appears evident (Kieran, 1992, p. 397). On the contrary, passing from the numerical to the algebraic context, a structural conception becomes crucial, in particular a key role seems to be played ”by the shift from a procedural view to a relational view of equality” (Carpenter & Franke, 2001, p.156). In other terms, algebraic *transformational activities* has to be based on the idea of changing the form of an expression, maintaining equivalencies (Kieran, 2003, p. 123).

This becomes crucial when transformation concerns algebraic rather than numerical expressions. The meaning of ‘transformation of expressions’ has to change; in fact, the final product of an algebraic transformation is not a number, neither one single expression, rather it is a new valid equivalence, relating two algebraic expressions. Within the numerical context, the basic properties of operations do not play an operative role; they simply express the equivalence of computing procedures, but they are not necessary and thus not usually employed for numerical computation.

On the contrary, within the algebraic context, operation properties must assume an operative role and must become the instruments for transforming expressions.

Actually, passing from numerical to algebraic calculation requires a crucial ‘revolution’: the properties of the operations have to overcome their status of “true” statement and become *rules of transformation*, i.e. "tools" of algebraic calculation. In order to accomplish this new status, they must assume a dual meaning: they must be properties stating the basic equivalence relations between algebraic expressions and tools of transformation, i.e. means through which equivalences between expressions are validated.

In other terms, equivalences stating the basic properties of addition and multiplication may function as an Axiomatic system for a local Algebra theory, within which a sequence of algebraic transformations (symbolic calculation) can be interpreted as a proving process, validating the equivalence between two algebraic expressions [2].

It is out of the scope of this lecture to fully discuss the reasons why such a theoretical approach may constitute an effective alternative to the traditional approach, and in what sense we got evidence of its success. I would rather explain, how we tried to implement this specific theoretical perspective exploiting the mediating potentialities of a microworld, but in order to do that, it is necessary to further elaborate on what is meant for *theoretical perspective*.

The notion of Theorem

Although the term proof is commonly used in the field of mathematics education, it is useful to remind that this term only partially describe the complex system of elements involved in what is to be considered a mathematical proving process. In fact, proof is only one of the components of a proving process, of course, there is a Statement to be proved and there is a theoretical system, a Theory, within which the proof makes sense. The notion of Theorem as the unity of statement, proof and theory, has been introduce to describe this complexity (Mariotti et al., 1997). Consider the case of Algebra. As explained in the previous section, given two algebraic formulas one can state their equivalence (Statement), prove its validity through a chain of transformations (Proof), each of them based on the application of one of the equivalence axioms assumed at the beginning, or one of the statement already proved (Theory) [3].

It is important to remark that what have been shortly referred to as Theory, has a twofold component. On the one hand, Axioms and already proved Theorems constitute the means of supporting the single steps of a proof; on the other hand, meta-theoretical rules assure the reliability of the specific way of producing an argument, in other terms, how Axioms and Theorems, belonging to a Theory, can be used to validate a new statement. As clearly pointed out by Sierpinska,

“[T]heoretical thinking is not about techniques or procedure for well-defined actions, […] theoretical thinking is reflective in that it does not take such techniques for granted but considers them always open to questioning and change. […] Theoretical thinking asks not only, *Is this statement true*? but also *What is the validity of our methods of verifying that it is true?* Thus theoretical thinking always takes a distance towards its own results. […] theoretical thinking is thinking where thought and its object belong to distinct planes of action”.

(Sierpinska, 2005, pp. 121-23)

According to school practice, the complexity of getting this meta-theoretical level seems to be ignored [4]. Grasping the functioning of a theoretical system is taken for grant; in particular, deduction rules and their functioning in the development of a Theory are rarely made explicit and consider as subject of discussion, on the contrary they usually remain completely under the control of the teacher, resulting in a general feeling of confusion and incertitude for students.

There are at least two aspects characterizing acting at a meta- theoretical level; one is related to the acceptability of some specific deductive means, the other to the fact that no other means, except those explicitly shared, is acceptable. Controlling these two aspects allows students to access to the meta-theoretical level, and consequently to get a theoretical perspective.

In summary, according to this analysis, a main direction emerges for the design of our microworld: the need of an environment where the use of specific tools could contribute to the evolution of meanings related to the notion of Theorem, as the unity of the three components (Statement, Proof and Theory), but in particular of tools contributing to make explicit and control both aspects characterizing the meta-theoretical level. In the following, a short description of L'Algebrista is provided, before focussing on a specific tool (Teorematore) and its way of functioning in respect to the development of a meta-theoretical perspective.

Microworld and theory

Starting from the previous analysis, a microworld was designed for Algebra Theory. Without going into technical details, I will focus on the general principles inspiring the design, that is I will explain how the epistemological and cognitive analysis have indicated key features required to make semiotic mediation possible.

The complex relationship among the different components of Algebra Theory has to be reproduced in a consistent way, so that acting on the objects in the microworld might generate meanings that could be related to the notion of Algebra Theory and to Theory in general.

As previously discussed, a formula stating the equivalence between two algebraic expressions constitute a *Statement* of our Theory, while the substitution of an expression with an equivalent one constitute the basic rule of transformation, through which one equivalence may be derived from another. The equivalences, stating the basic properties of addition and multiplication, constitute the basic set of Axioms to start with [5], while the substitution process constitutes the deduction rule in play within this Algebra Theory.
As a consequence, besides the counterpart of the Axioms as statement, we also need a counterpart of the substitution process; in other terms, we need objects referring to *any single axiom and to its application*, that is its functioning in a proof.

According to this analysis, a microworld was designed (L'Algebrista) whose elements and their functioning can be related to the main elements of Algebra Theory as follows:

Axioms as statements ←→ Commands Buttons, represented on the screen by icons.

Deduction rule ←→ Clicking on any icon activates the corresponding command, which acts on a formula, previously selected, substituting it by the corresponding equivalent one.

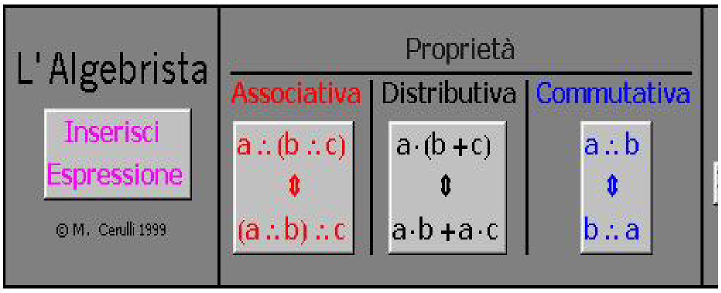

Figure 1: he icons representing the properties are also active buttons, each of them corresponds to the statement of an axiom and it functions as tool according the basic deductive rule.

It is important to remark that the different components of the notion of theory find a counterpart in the Buttons and their functioning (i.e. in the commands available) within the microworld. In this sense the whole Palette, grouping the available buttons, provides a counterpart of the Axiomatic. In order to stress its specificity, the constraint of acting within a theoretical domain is explicitly represented by the action of "entering" the microworld, which is accomplished by using the command Insert Expression (Ita. Inserisci Espressione). Such a command has the effect of initializing the application, that is, it changes the status of the selected expression, which only in this way becomes an object of the microworld.

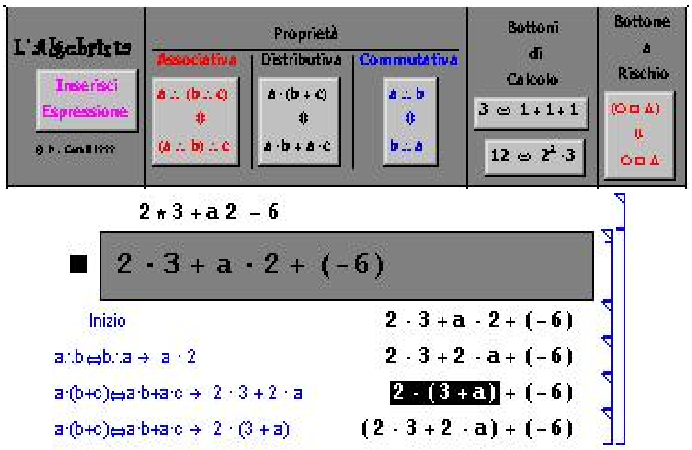

*Figure 2: The user writes the expression to work with («2 * 3 + a 2 - 6» in the example), then after selecting it the button Inserisci Espressione is clicked, thus L'Algebrista creates a new working area where the buttons are active.*

Figure 2 shows an example of this procedure in the case of the axiom of Commutative property of addition and the Distributive property. Note that the action of any property button is reversible, as any equivalence has to be.

Experiencing deductive rules

Acting in the microworld offers the opportunity of active experiencing both the acceptance and the constraints of the deductive rules in play.

In fact, the functioning of the microworld has a direct counterpart in the functioning of the theoretical environment, and accepting the constraints of the microworld has a direct counterpart in accepting the functioning of a theoretical framework. On the one hand, the limitation defined by the buttons available corresponds to the limitation defined by the axioms available. On the other hand, the action of the commands corresponds to the deductive rule of substitution.

Moreover, the very presence of the Axiom/Buttons offers the opportunity of objectifying (Radford, 2003) these ideas and consequently allows the emergence of a reflective discourse about their functioning. Given this key of access to a theoretical perspective, its evolution may progress under the guidance of the teacher.

More precisely, in the vygotskian framework of semiotic mediation, one can state the following hypothesis:

Specific elements of the microworld can be interpreted as signs, referring to mathematical meanings, and as such the can be exploited by the teacher according to specific educational aims.

- *Expressions in L'Algebrista* “are” signs of *algebraic expressions*
- *Buttons/icons* “are” signs of *axiom statements* and *definition statements of an Algebra Theory*;
- *Buttons commands* “are” signs of *application of axioms according the basic deductive rule.*
- *Transforming an expression*, *using the available commands* “is” a sign of proving *within the stated Algebra Theory...*

This correspondence has to be exploited through semiotic activities (collective discussions, but also writing reports or just sharing the solution of a task) orchestrated by the teacher. The aim is that of making signs emerge and meanings evolve. In particular, when a new equivalence is achieved in the microworld, this fact can be interpreted as the fact that a new statement has been proved in the Algebra theory that is a new Theorem can be added to the set of the available theorems. A careful didactic analysis has to be carried out in order to design such activities.

Example of deduction

A fine grain analysis of the functioning of the process of semiotic mediation is out of the scope of this contribution. Some results, related to specific teaching experiments based on the use of L’Algebrista can be found in (Mariotti & Cerulli, 2001; Cerulli & Mariotti, 2002; Cerulli & Mariotti, 2003).

Figure 3 shows an exemplar of students’ solution to a comparison task, where students are asked to evaluate the equivalence between different expressions. The student firstly checks the equivalence using his computing skills, once he made his conjecture he uses the properties of the operations (i.e. Axioms) and a theorem to prove it. A translation of each statement is reported on the left of the image.

I think the 1st and the 3rd are equivalent, but not the 2nd, because applying the properties they become equal, while the 2nd does not.

I applied the distributive property.

I applied the distributive property on these two pieces.
I added the two equal terms **–a*b –a*b** and I cancelled its result with its opposite obtaining **0** for the **1st theorem**.
I cancelled also **+b*b** with its opposite and as it was **–2b*b** I obtained **–b*b.**
At this point the 3rd expression is equal to the 1st expression.

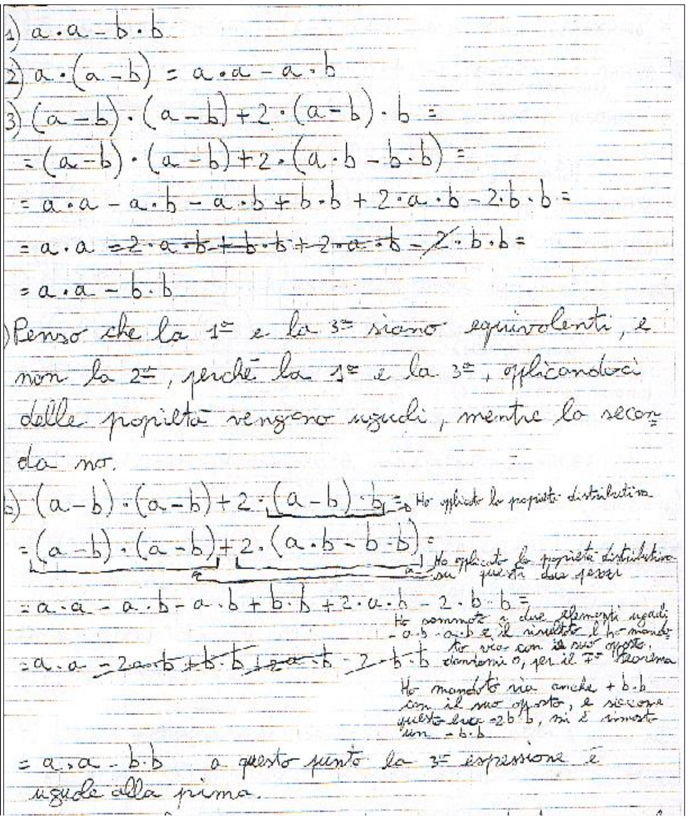

Figure 3: Exemplar of student's solution of a comparison task

Development of the theory

According to our hypothesis, and as clearly appears in the following protocol, the students may grasp the new status of an equivalence, as soon as it has been proved. Never the less, the act of enlarging of the theory by assuming new means of proving constitutes a delicate point in the development of a theoretical perspective; that is it

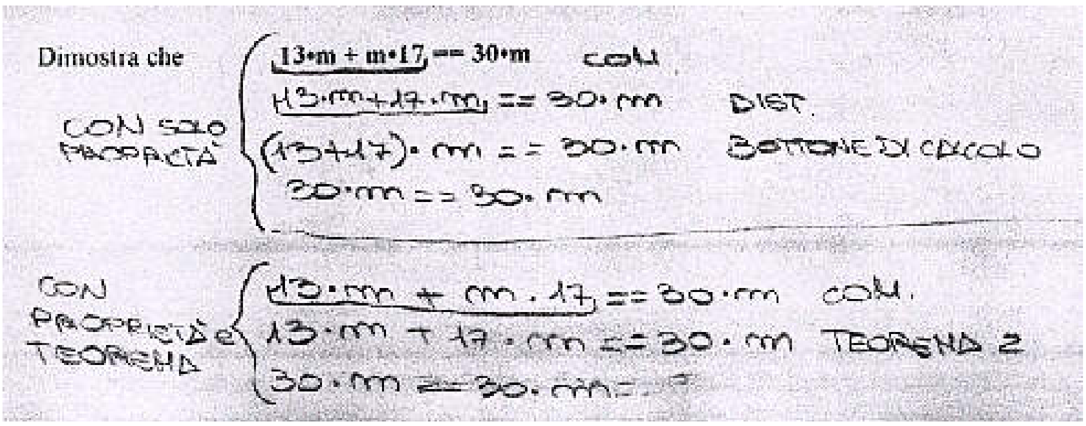

Figure 4: The student provides two solutions and compare them. The first proof is characterized as "only with Properties", the second as "with Properties and Theorem"

may be difficult to accept that a new statement can be applied in the same manner as the axioms available.

In this sense, the protocol shows an interesting case: although not required, Marta gives two different proofs of an equivalence: the first referring to the basic set of axioms, the second referring to the enlarged theory, including what she calls Theorem 2. The difference between the two transformation chains has been clearly understood; moreover one can say that the student takes a meta-theoretical perspective when she explains the difference between the two proofs in terms of the difference between the two Theories of reference (as she says: “only with properties” or “with properties and theorem”). This behaviour reflects the experience Marta had within the microworld.

In fact the microworld offered the following facilities:

- *new buttons* might be created and “become” signs of *theorems as available means to produce proofs of new statements, i.e. theorems;*
- *a tool realizing the introduction of new buttons*, it becomes “is” a sign of the meta-theoretical operation of *adding new theorems* to a theory.

That was accomplished by the design of a particular environment, the access to which is achieved entering a new menu: the Meta menu.

The Meta menu

As the word *meta* evokes, this menu was designed to offers tools to be used to act on the set of the available Buttons, that is to the set of buttons corresponding to the Theory itself [6].

Figure 5: The Meta menu.

The first tool is called **Theorem maker** (Ita. ***Il Teorematore***), it activates an environment where the user can create new buttons to represent new transformation rules. The new buttons can be included in palettes and used to transform expressions in L'Algebrista.

The second tool is called **Palette personalizing** (Ita. ***Personalizza Palette***), it is just a notebook containing a collection of ready-made Theorem/buttons and instructions concerning how to create a palette using those buttons and any button created with *Il Teorematore*.

The Theorem maker

The Theorem maker provides an environment that is a counterpart of the meta-theoretical level. Let us analyse its functioning in terms of its potentialities in semiotic mediation process.

Once an equivalence relationship is proved, two changes occur:

- a statement changes its status, attaining the **status of theorem** that can be used to prove new statements

- the theory, as set of statements that can be used, changes, it acquires a new element.

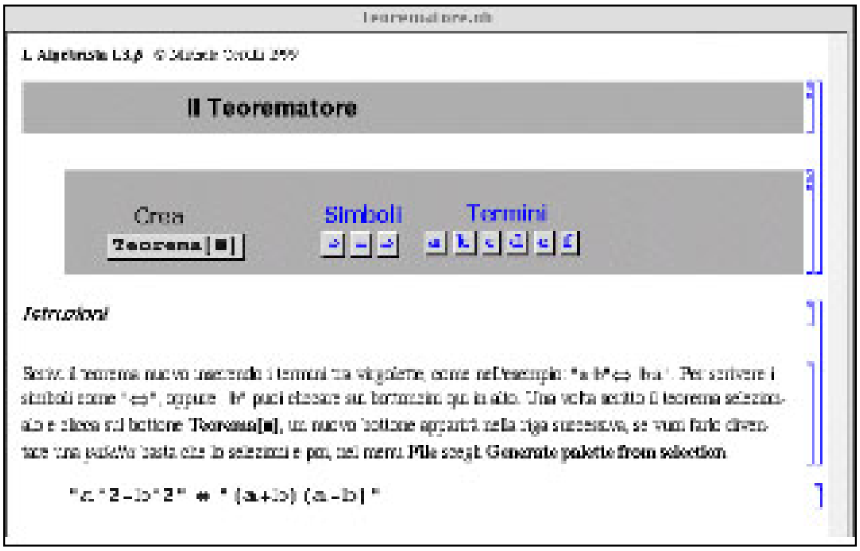

Figure 6: The Theorem maker environment.

The design of the tools aimed to provide a counterpart to the act of changing both of the status of a statement and the power of the theory.

In order to change the status of a statement, when a new button is created, the user has to re-write the statement, according to specific constraints of the Theorem maker.

Attaining the status of **theorem** corresponds to the fact that the statement in focus has to overcome the context in which it was produced and getting a higher generality that allows its use in the substitution process. This is a very delicate point: actually such a generality requires that the domain of interpretation of a letter must be extended to the whole domain of algebraic expressions. The possibility of different levels of interpretation find a counterpart in the functioning of the editor of the Theorem maker, in particular in the use of different fonts in the editing of the statement new button, but also in the use of Letter/buttons that must be activated. In fact, though it is possible to edit a button using the letters of a regular notebook, in order to make the new button usable as a Theorem in a transformation chain, only the use of specific Letter/buttons is effective.

The use of regular letter has the effect that the statement will have a validity limited to itself: no substitution except those concerning the specific letters involved will be possible.

On the contrary, clicking on a Letter/button in the editing of the new Theorem/button corresponds to assign the needed generality to the variables of the statement. In other terms, using the *Theorem maker*, it is possible to create a new button, which incorporates the proved equivalence relationship at the highest level of generality and the very act of assigning such generality is represented by the use of special editing buttons, that may become ”signs” of the general status of letters in the substitution process.

Moreover, once created, the new buttons can be grouped and placed within a particular Palette, expressly defined by the user with the command Palette personalizing (see figure 7 below). This represents the counterpart of changing the status of the theory. But this is accomplished so that the difference between the basic assumption and the new acquisitions remains evident. In fact, it is possible to create different palettes and put any button in any palette, except in any pre-defined "*Teoria#*" palette. Consistently, because of their reference to specific sets of axioms, those palettes cannot be modified; in other words, the location of a button in L'Algebrista, incorporates its status in terms of being an axiom or being a theorem.

Figure 7: A palette of theorem buttons, as created by our pupils during a teaching experiment.

Thus, such organization of the palettes of buttons and the constraints of its functioning corresponds to a classification of the statements according to their status in an algebraic theory.

Conclusions

As said in the introduction, our design work has been directed by the articulation of different kinds of analysis. Previous discussion has shown the articulation between at least two kinds of analysis: the epistemological analysis, related to the mathematical knowledge, which is the object of the educational intervention, and the cognitive analysis, related to the hypothetical artefact that is to be designed. Because of the constraints given to this paper the third direction of analysis, the didactic analysis, was not treated. Actually, some elements of the didactic analysis emerged implicitly from the discussion but it was not possible to provide a full explanation.

The general assumption, underlining the previous integrated analysis, concerns the use of an artefact as potential tool of semiotic mediation (Vygotsky, 1978, Mariotti, 2002), that means that the design of the microworld had a main inspiring idea: create computational objects the use of which might be directly related to mathematical meanings. Previous experience with a particular Dynamic Geometry Environment had shown the potentialities of a microworld in introducing pupils to a theoretical perspective (Mariotti, 2001a). In particular, the instrumental aspect of Axioms in the production of a proof has been related to the use of specific commands that could be semantically related to them. Analogously, in L'Algebrista the parallel between commands and axioms has been exploited, so that acting through commands in the microworld could have a direct counterpart in proving within a theory. In this sense the notion of Theorem as the triplet (statement, proof, theory), find a semantic domain in the domain of "transformations of expressions in L'Algebrista". Furthermore, the specific environment was designed to offer mediation tools for key aspects of acting at the meta-theoretical level; that means to offer elements, which may become tools of semiotic mediation to introduce pupils to complex ideas such as that of theoretical status of a statement or that of enlarging a Theory by adding new Theorems.

Showing the success of our choices was not in focus in this paper, rather the methodology used in the design process; in particular I want to stress some aspects, which in my opinion are related to the basic assumption concerning the process of semiotic mediation.

When an artefact is designed the main aim is to make it produce, as better as possible, a specific product, for instance a drawing, if it is the case of a graphic environment, or the result of a symbolic manipulation, if it is the case of a symbolic manipulator. On the contrary, when an artefact is designed with the objective of becoming a potential tool of semiotic mediation certainly the efficiency in the manufacture may lose its centrality and different criteria of analysis have to be assumed. For instance the dialectics between action and constraints, which plays a crucial role in the construction of schemes of utilization of an artefact and consequently in the construction of meanings emerging from it use, must have a direct counterpart in the criteria for selecting what is automatic and what is left under the control of the user, what is controlled by the environment and what is left without any control.

The method used in our design, has been strongly based on the interplay between epistemological and cognitive aspects, further investigations are required to refine the criteria to be used in the design, and the didactic component has to come to the forefront.

Endnotes

[1]: The term artefact referring to any generic product of human culture, purposefully designed to act or interact in a human setting. I follow the distinction, introduced by Rabardel (1995), although I will not exploit all its potentialities.

[2]: A full discussion can be found in (Cerulli, 2004).

[3]: The last component of the proof, i.e. the Theory within which the proof make sense, is largely neglected and a part from well known exceptions, like Geometry, the theoretical context within which theorems are proved is often left implicit. This is often the case, for instance, in Calculus theorems.

[4]: An exception is that of mathematical induction, which is explicitly treated, and to which is devoted a specific training. But, very rarely, mathematical induction is presented in comparison with other modalities of proving, which are commonly considered natural and spontaneous way of reasoning.

[5]: For instance the statement “a+b=b+a”. For brevity reasons, I will not enter into details in the description of axioms and definitions of the Theory, I would rather concentrate on the meta-theoretical aspects, which I am interested in.

[6]: Recall that with theory we mean set of axioms, definition and theorems represented by buttons.

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