Learning mathematics at university level : what can we learn from recent developments of didactic research

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Abstract

In this text, I first evoke discussions developed in the educational community about the specificity of mathematical learning at university level or about the nature of avanced mathematical thinking. Then, to reflect on what is offered by recent didactic research carried out at university level, I focus on three dimensions. In my opinion, these dimensions evidence the potential offered by the evolution in research approaches and interests for setting up more adequately the questions at stake, and for progressing in the knowledge of learning processes. These are the followings:

- *the increasing attention paid to flexibility in the study of learning processes;*
- *the evolution of research paradigms from constructivist approaches towards anthropological and socio-cultural approaches;*
- *the extension of research domains.*

I. Introduction

I.1. Learning mathematics at university level and advanced mathematical thinking

Reflecting on the learning of mathematics at university level, a first question emerges. Is mathematical learning at this advanced level something different from what is mathematical learning at primary or secondary levels? What are its specificities if any?

Such a question was for instance extensively discussed in the working group *Advanced Mathematical Thinking* (AMT in the following) of PME [1], in the late eighties. Nevertheless, in the book that emerged from this working group (Tall, 1991), the question is not really answered. In the first chapter of the book titled : *The psychology of advanced mathematical thinking*, D. Tall writes:

> « The move from elementary to advanced mathematical thinking involves a significant transition : that from *describing* to *defining*, from *convincing* to *proving* in a logical manner based on those definitions. This transition requires a cognitive reconstruction which is seen during the university students' initial struggle with formal abstractions as they tackle the first year of university. It is the transition from the *coherence* of elementary mathematics to the *consequence* of advanced mathematics, based on abstract entities which the individual must construct through deductions and formal definitions. » (p.20)

But, in the second chapter of the same book titled: *Advanced mathematical thinking processes*, T. Dreyfus, for his part, points out that characterizing AMT is not an easy task, as the same processes : representing, visualizing, generalizing, categorizing, conjecturing, deducing, analysing, synthesizing, abstracting and formalizing … can be found at any level :

> « There is no sharp distinction between many of the processes of elementary and advanced mathematical thinking, even though advanced mathematics is more focussed on the abstractions of definition and deduction. Many of the processes to be considered in this chapter are present already in children thinking about elementary mathematics concepts, say number or place value. They are not exclusively used in advanced mathematics, nor, indeed, are they exclusively used in mathematics. Abstractions are made in physics, representations are used in psychology, analysis is used in economics and vizualisation in art. » (p. 26)

According to him, what can make the difference between advanced mathematical thinking and elementary mathematical thinking is complexity and how this complexity is dealt with. Such a perspective leads him to pay a specific attention to abstraction and representation processes in the chapter:

> « The distinction is in how this complexity is managed. The powerful processes are those that allow one to do this, in particular abstracting and representing. By means of abstracting and representing, one can move from one level of detail to another and thus manage the complexity. »

This point of view can be related to the perspective developed by A. Robert in the special issue of the journal *Recherches en Didactique des Mathématiques* devoted to the didactics of mathematics at post-compulsory level, following a specific work on this theme at the 9th Summer School in the Didactics of Mathematics, in France (Robert, 1998). She first stresses the complexity of the mathematics at play at post-compulsory level, and the changes this complexity induces in the mathematical practices expected from students, even if such practices are not completely new. She organizes the description of these changes around three dimensions: proofs, knowledge forms and levels of use, personal work, and she defines several descriptive criteria for each of these three dimensions. From this study emerge then four dimensions of analysis for the mathematical contents to be taught which also support the design of teaching scenarios.

From 1998, a working group of PME-NA [2] titled : *The Role of Advanced Mathematical Thinking in Mathematics Education Reform* also focused on these questions, in connection with the new NCTM Standards (NCTM, 2000). The group aim was to reflect on the kind of mathematical experiences that could help the student transition towards the forms of AMT that post-secondary education often requires from its students. The results of this reflection have been recently published in a special issue of the journal *Mathematics Teaching and Learning* coordinated by J. and A. Selden (A. Selden & J. Selden, 2005). The reader finds there the same duality as the one mentioned above with, on the one hand, new attempts made for characterizing AMT as something different from EMT (attempts which, in my opinion, do not lead to fully convincing definitions), and on the other hand the accent put on the continuity which can be a priori offered by mathematical pratices from childhood to adult learners, and the interest, from a didactic point of view, of thinking in terms of evolution and development instead of in terms of opposition and gap. The title of some papers reflects this perspective: Advanced Mathematical-Thinking at Any Age : Its Nature and Its Development (Harel & Sowder, 2005) or Advancing Mathematical Activity : A Practice-Oriented View of Advanced Mathematical Thinking (Rasmussen, Zandieh, King, Teppo, 2005). As is stressed by these authors, using the terms advancing and *activity* instead of *advanced* and *thinking* is not at all neutral :

> « We use the term *advancing* (versus *advanced*) because we emphazise the progression and evolution of students' reasoning in relation to their previous activity. We also use the term activity, rather than thinking. This shift in language reflects our characterization of

progression in mathematical thinking as acts of participation in a variety of different socially and culturally situated mathematical practices. »

Even if I think important to mention the past and current discussions about the nature of AMT, I don't want to engage here in a debate on this topic. First of all, because I do not see this question as the most crucial one today. When preparing this text, what seemed to me especially important, was to consider mathematical learning at university level in a very open way, avoiding the trap of limiting the perspective to the forms of thinking and learning with which I am the most familiar, due to my institutional position and my personal mathematical practices. In other words, what I am interested in is the learning of mathematics at university level with the diversity of its facets, considering the future mathematicians or the big majority of those whose professional relationships with mathematics will be more limited, considering the access to formal mathematics but also the satisfaction of the diverse mathematical needs of different professional categories. If one accepts this perspective, questioning learning processes at university level can be especially interesting and productive because, on the one hand these learning processes do not aim only at developing some kind of general mathematical culture, on the other hand because university students, whatever be their initial background, have generally sophisticated mathematics to learn. This reflection can thus be also insightful for those who work at more elementary levels of schooling, and for instance help us consider the particular issues raised by vocational training, so marginally addressed in didactic research up to now (Straesser, 2005).

I.2. A tradition of research and some more recent evolutions apparently promising

I have evoked above the book *Advanced Mathematical Thinking*, the first reference book published in that area of research. Its reading is really insightful for those who want to understand how research began to develop at university level, what issues researchers first focused on, how they approached these according to their didactic culture, the theoretical constructs and experimental results their research led to... But what mainly inspired this paper, beyond my own research in the last ten years and that of some students of mine, is the synthetic work carried out with A. Schoenfeld when coordinating the chapter on research in the book resulting from the ICMI Study on the teaching and learning of mathematics at university level (Holton, 2001), and also the chapter I have just written in collaboration with C. Batanero and P. Kent for the second NCTM Handbook on research in mathematics education, coordinated by E. Silver [3].

As evidenced both by the AMT book and by the different research synthesis published in the ICMI Study Book (Artigue, 2001), (Dorier & Sierpinska, 2001), (Robert & Speer, 2001), (Selden & Selden, 2001), research on mathematical learning at university level focused first on the basic undergraduate courses: courses in Calculus or elementary Analysis, in algebra and linear algebra. Researchers tried to better understand the cognitive difficulties met by the students – these difficulties being seen as the main source of the important failure observed in undergraduate courses - and, from this understanding, to elaborate more efficient teaching strategies. Research firstly pointed out the existing gap between the logics underlying the mathematical constructions presented to the students - a logics achieved through a long and complex history - and the logics underlying the students' cognitive development. This perspective led the researchers to introduce specific constructs for labelling some important oppositions and dualities. They thus distinguished concept *definitions* and *concept images* (Tall & Vinner, 1981), *processes* and *objects* (Dubinsky, 1991), *structural* and *operational* visions of mathematical concepts (Sfard, 1991). They also associated with these distinctions cognitive hierarchies such as the hierarchy between *action, process* and *object* at the source of the APOS theory (Dubinsly & Thomas, 2001), or the rather close constructions one can find in the initial research work by Tall and Sfard. Another notion used in order to understand the students' difficulties was that of *epistemological obstacle* and the research developed by A. Sierpinska about limits and functions is certainly the best illustration of this approach (see for instance (Sierpinska, 1994) for a synthetic vision). What seems to me important to point out is that, whatever be the theoretical frames founding them, for understanding and overcoming teaching and learning difficulties at university level, these research works, in their great majority, started from the analysis of students' conceptions and difficulties, and doing so focused on the discontinuities of learning processes. This of course had evident influence on the didactic strategies and designs built in order to support the mathematical development of students. The situations proposed to students had to be the source of cognitive conflicts whose resolution would allow them to overcome misconceptions or epistemological obstacles, or to organize the students' progression along the identified hierarchies, such as for instance the action-process-object hierarchy.

In this text, I will not come back on the results of the research carried out within these perspectives. These are now well known. I would like to focus on some more recent research perspectives. Even when they address the same issues, they approach these in a renewed way. They also raise new issues because they consider other educational contexts, other mathematical domains than those extensively explored up to now. For these reasons, they seem to me especially interesting, obliging us to question positions on mathematical learning that perhaps have naturalized in educational communities and taken the status of evidence too quickly. In the diversity of the research work which could support this aim, I have decided to favour three directions [4]:

- the transition from perspectives focusing on cognitive discontinuities and hierarchies towards perspectives more sensitive to issues of flexibility and connection;
- the new approach on institutional transitions (from secondary to tertiary here) made possible by the evolution of the theoretical approaches from constructivist paradigms towards socio-cultural or anthropological paradigms;
- and, finally, the new perspectives offered by the development of research in less standard educational contexts, such as the training of engineers.

II. The increasing sensitiveness towards cognitive flexibility and connections

This increasing sensitiveness allows a more balanced vision between what can be thought as a vertical process of conceptualization by abstraction, generalization development of more and more general structures, and an horizontal process of conceptualization relying on connection between contexts, mathematical domains, perspectives on mathematical objects, semiotic representations of these. Of course, this second aspect was not missing in didactical concerns and constructs. Think for instance about the crucial role played by the interplay between *settings* in the theoretical frame developed by R. Douady from the late seventies, and known as the *tool-object dialectics* (Douady, 1984) [5]. In this approach, the interplay between settings is seen as a privileged lever for making possible the emergence of new knowledge from the old one. Through the adequate conversion of problems raised in one setting into another setting, students, similarly to mathematicians, are provided with new mathematical tools allowing them to progress and to reach states of knowledge not directly accessible in the initial setting.

Nevertheless flexibility has been given much more attention in the last 15 years. This increase in attention has been supported by an evolution of the theoretical frames towards perspectives giving an essential role to the semiotic dimension of learning processes (cf. as an example (Duval, 1995)), and by the technological evolution. The latter introduced new modes of representations for mathematical objects, new possibilities for connecting representations productively, and favoured the modelling of knowledge in terms of networks (Hoyles & Noss, 2003).

Illustrating these changes by looking at the evolution of research in the domain of functions would certainly have been the easiest choice: the vision we have of the conceptualization of this notion has been so deeply affected by these changes! Nevertheless, I have chosen another domain whose approach has certainly been less influenced by the technological evolution but which seems to me especially interesting: linear algebra.

In the development of research in this area, as shown in (Dorier, 2000) and (Dorier & Sierpinska, 2001), what has been first pointed out is the high level of abstraction of the basic concepts of the field such as the concept of vectorial space, and the resulting difficulties for university students. This led to question the usual teaching strategies imposing from the start, without any motivation, the formal definition of the structure of vectorial space, strategies blind to the fact that this definition was the product of a long and complex history and that its interest did not impose immediately to the mathematicians themselves once it was introduced by Peano in 1880.

But linear algebra is also a domain where learning difficulties are linked to the diversity of the flexibilities at stake. From this point of view, research such as those carried out by A.Sierpinska and J.Hillel in Canada or K. Pavlopoulou and M. Alves Dias in France, also reported in (Dorier, 2000) are especially interesting. Linear algebra is, as stressed by Dorier:

> « an ‘explosive compound’ of languages, settings and systems of representation. There is the geometric language of lines and planes, the algebraic language of linear equations, n-tuples and matrices, the abstract language of vector spaces and linear transformations. There are the settings of geometry, of algebra, but also of graphical representations which allow a metaphoric use of geometry in higher dimensional spaces. There are the ‘graphical’, the ‘tabular’ and the ‘symbolic’ registers of the languages of linear algebra. » (p. 274)

There is also the interplay between cartesian and parametric point of views, which can be managed algorithmically in finite dimensional spaces and functions as a metaphor in infinite dimensional spaces. To this adds the diversity of the reasoning modes. A. Sierpinska, for instance, relying on an historical analysis, identifies three different reasoning modes which strongly intertwine in linear algebra: the *synthetic and geometric mode* where mathematical objects are, in some way, directly given to the mind, which tries to grasp and describe them; the *analytic-arithmetic* mode where objects are given indirectly by formulas or equations which make it possible to calculate with them; and the *analytic-structural* mode where objects are also given indirectly, but this time through a set of properties. She points out that :

> « While these modes of thinking appeared in the history of mathematics in a sequential manner, it did not happen that one of them eliminated the other two. The development of algebra owes a lot to their constant interaction; from the analytic geometry of Descartes to the geometric intuitions underlying the theory of Banach spaces. Indeed the story of linear algebra can be told so as to illustrate this continuous activity of change in perspective, done in the hope of gaining a better and deeper understanding of the domain. »

But, what her research shows, as the other mentioned above, is the complexity at stake in these changes in perspective, and in the interplay between languages and representations associated with these. It is the difficulty students meet at developing the corresponding competences. It is also the limited attention that university teachers pay to the development of this essential component of the power of linear algebra, as if it were a simple by-product of conceptualization and not a fundamental component of it. Observations made of teaching and tutoring practices show that teachers permanently jump from one language to another one, from one system of representation to another one, using the one for controlling or interpreting the mathematical work performed with the other, without any particular precaution, as if it were easy. But it is far from being easy.

As much as the progression towards increasing levels of abstraction, the progression in connection and flexibility between contexts, settings, semiotic registers, perspectives… is something essential to mathematical learning, and something difficult that needs to be explicitely taken in charge by teaching and planned as a long term process.

III. The move from constructivist towards socio-cultural and anthropological approaches

This move, which is an international phenomenon, has led to different theoretical constructions according to the didactic cultures (Lermann & Sierpinska, 1996) but what these constructions share is the basic assumption that mathematical objects are not universals but entities emerging from human practices, these practices being institutional or socio-cultural practices and thus dependent on tools, contexts and cultures. The relationships we develop with particular mathematical objects or domains are thus shaped by the practices existing in the institutions where we meet them, and the norms and values that these institutions associate with them. Within such a perspective, in order to understand individual learning processes, one has first to know the institutional practices that shape these processes, determine what they make possible to learn and what they make difficult if not impossible to learn. The learning processes we observe are only those that the functioning of the institutions where they take place allows us to meet. Complexity, in some sense, and focus move from the cognising subject to his(her) institutional and cultural environment.

In order to illustrate the potential offered by this change in perspectives, I will take an example, that of the transition between high school and university. This transition is more and more percieved as a problematic one, for reasons I cannot detail here. These have been carefully analysed in the ICMI Study mentioned above (Holton, 2001), the problems linked to this transition having been one of the main reasons for launching this Study. University teachers all around the world more and more complain about the mathematical knowledge of incoming students, and blame secondary curricula and teachers. Socio-cultural and anthropological approaches lead us to analyse this transition as a transition between cultures and to investigate what are their respective characteristics, what distance exactly separates these, what the secondary and tertiary institutions do in order to help the transition, what they let to the private responsibility of students, and what are the cognitive effects of these institutional behaviours.

These approaches also generally consider that mathematical cultures, as any kind of culture, have different components, only a small part of these being made explicit, what makes any kind of cultural transition a problematic one. Investigating the characteristics of secondary or tertiary cultures, in their explicit and implicit dimensions, is thus a complex work but a necessary work for anyone who wants to understand the real difficulties of the secondary-tertiary transition and how these could be overcome. In order to illustrate the potential offered by such approaches, I will use the thesis by a student of mine: F. Praslon (Praslon, 2000) who focused on these transition issues as regard the notion of derivative and its mathematical environment. The analysis he has developed, thanks to the triangulation of diverse sources of data, shows for instance that, conversely to what is often said:

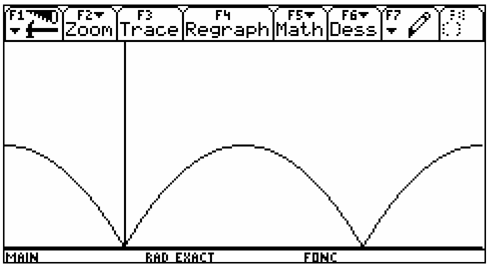
- what is expected at the end of high school in France as regard the notion of derivative constitutes already an important corpus of knowledge (the conceptual map he draws from these expectations covers two A4 pages),
- the secondary-tertiary transition in this area is not a transition between informal and formal mathematics, between intuitive and rigorous approaches.

In fact, University has already tried to adapt to the evolution of its population and the transition is today better described as an accumulation of “micro-breaches”, which affect different dimensions of the students’ mathematical activity. This accumulation induces a real cognitive gap to which university teachers are less sensitive than would be the case if there was a clear rupture from the informal to the formal. The multidimensional grid of analysis designed by F. Praslon allows him to identify and describe these micro-breaches. Let me mention some of these:

- an increased rate in the introduction of new objects requiring from students quicker assimilation and accomodation;
- an increased diversity of tasks to face, which makes routinization much more difficult to achieve;
- an increased autonomy given to students as regard the choice of appropriate settings, appropriate semiotic registers, and the connections and changes made between these during the solving process, and more globally as regard the overall development of the solving process (similar tasks often appear in the two institutions, but the number of intermediate questions in a problem is far from being the same);
- a new balance between the study of particular objects and objects defined by general conditions, which gives another role to properties and theorems. It is no longer sufficient to memorize properties and theorems in order to invoke them in standard particular cases. Students have also to reflect on their conditions of validity and, for that purpose, beyond the standard cases they are familiar with, look for non trivial situations that could invalidate them;
- theorems more systematically proved with proofs taking the status of methods; this is not the case at high school level where proofs do not really have an operational role in elementary analysis and can more be seen as “a cheery on a cake”, the most important being the theorem itself, a new piece in the student toolbag.

In order to make both students and university teachers sensitive to the differences between the two cultures, F. Praslon has created a set of tasks which, according to his results, are situated today in a gap between the two cultures, and reveal the main facets of the differences between these. Purposely, none of the tasks requires solution by formal analysis. They are designed to be proposed to the students before their entrance to university or at the beginning of the academic year, and the students' work discussed with university teachers. Below is an example of such a task.

One consider the function f of one real variable, periodic with period 1, defined by: $f(x)=x.(1-x)$ on the interval [0, 1[(a graphical representation similar to the one below is given)



Q1 : Is this function continuous ? Does it have a derivative ?

Q2 : One considers a new notion, called the symmetric derivative and defined as the limit, if it exists, of the symmetric rate of increase $(f(x+h)-f(x-h))/h$ when h tends towards 0. Compute the derivative and symmetric derivatives of f (il they exist) for x successively equal to ½, ¼ et 0, and compare their values.

Q3 : Below are three conjectures. Are they true or false ? Justify carefully your answers.

C1. Every even function defined on IR has a symmetric derivative at x=0.

C2. Every even function defined on IR has a derivative at x=0.

C3. If a function defined on IR has a derivative at x=a, it has also a symmetric derivative at x=a and the two are equal.

A priori this task can be solved with the French high school mathematical knowledge. Nevertheless it is not part of the French high school culture for several reasons. The function f is defined by parts, and the student has to understand that the given expression only works on the interval [0,1[. Such functions are marginal in the high school culture. Questions are raised about its continuity and derivability. These are not new questions, and their solving is supported by the graphical representation given. Nevertheless, such questions at high school level generally are raised about functions defined by only one formula, which are sums, products, quotients or simple compositions of functions known to be continuous or with a derivative. The techniques to be developed here, if the student wants to solve the problem analytically and not just graphically, are not the same. The question Q2 introduces a new notion through it formal definition and students have to work with this definition. This, also, is not an usual task, and the same can be said about the general conjectures proposed in Q3.

What are the answers given by the scientific students tested in the thesis at the beginning of their first university year?

When dealing with question Q1, about one third of the students don't see that there is a problem: the function is defined by a polynomial expression, hence is continuous and has a derivative on IR; some, more cautious, invoking the same reason, write that f is continuous and has a derivative on [0,1[, and that, being periodic with period 1, these properties are valid on IR. About one fourth identify the problem of derivability and try to justify their answer graphically or analytically. The necessity of developing a local study is more acknowledged for x=1 than for x=0. According to F. Praslon, this discrepancy is caused by the fact that the interval given in the definition of f is closed in 0 and open in 1. Their experience can have led students to attach this symbolic characteristic (open bracket) to problematic cases. The arguments provided by those solving the question graphically are more or less the following: « f does not have a derivative at x=0 because the curve representing f has two different half tangents » or « has a corner », « has an angular point ». In case of an analytic proof, the most elaborated arguments are of the following type: f has a derivative at x=0 (or x=1) if and only if f(0)=f(1) and f'(0)=f'(1); then, using the expression given for f: f(x)=x(1-x), they conclude that the second condition is not satisfied because f'(x)=-2x+1. Moreover, F. Praslon notices a clear separation between graphical and analytical answers.

Answers to question Q2, conversely to what could have been expected, show that the formal introduction of a new notion is not of source of block except for a minority of students. Computations of the derivative and symmetric derivative are quite successful when the computation only needs the correct instantiation of the formal definition, that is to say for x=½ and x=1/.4 (95% (derivative) and 75% (symmetric derivative) of correct answers). Nevertheless, for x=0, existence and computation are only correctly delt with in 14% and respectively 3% of the answers. Students use the expression given for f in the computation, and even when the obtained result is not coherent with their answer to question Q1, they do not see it. There are not many answers to question Q3, and when a counter-example is provided, generally the function quoted is the absolute value function and not the function f.

As shown by the doctoral thesis, beyond this particular exercise, students are not blocked when facing such unusual questions but they do not have efficient mathematical tools for tackling these. As was mentioned above, university teachers are generally poorly sensitive to the cognitive requirements of such tasks which can be solved without ε and η reasoning. They have difficulties in measuring the real distance separating the secondary and university cultures and in seeing how it could be progressively covered. Even when they try to help their students, this lack of knowledge make them rather inefficient.

M. Bosch, C. Fonseca et J. Gascón, in an on-going research project (Bosch, Fonseca, Gascón, 2004) address also the secondary-tertiary transition, but they adopt a perspective more radically anthropological than F. Praslon. Praslon, while situated globally in an anthropological perspective, used notions and categories of analysis coming from different approaches in order to benefit from the huge amount of results already obtained by research on Calculus. In the work by the Spanish colleagues, the de-personalization of the didactic analysis induced by the anthropological approach is much more strongly asserted through the opposition made by these researchers between what they call the *Cognitive Program* and the *Epistemological Program*. According to them, the first one considers that the phenomena related to the learning and teaching of mathematics, and especially the problematic ones, can be explained by individual characteristics of the learners, while the second one considers that the understanding of didactic phenomena has to be looked for in the mathematical practices of the educational institutions. Analysing these with the conceptual tools offered by the Chevallard's anthropological approach, they show the predominance at secondary level of pointwise or local mathematical organizations, both rigid and incomplete, focusing on the practical-technical side of praxeologies, while University tends to focus, from the start, on regional mathematical organizations, and on the technological-theoretical side of the corresponding praxeologies [6]. In this research also, questionnaires are designed and proposed to students but, as is stressed by the authors, they do not aim at analysing the students' mathematical knowledge:

> « Nuestro objetivo principal consiste en utilizar las respuestas de los estudiantes como indicadores de algunas de las características de las OM (Organizaciones Matemáticas) que se estudian en S (Secundaria) y poner de manifiesto la existencia y la naturaleza de determinados obstáculos epistemológicos y didácticos que dificultan el desarrollo del proceso de estudio de las matemáticas en el paso de Secundaria al primer ciclo de la Universidad. » (p.227)

Comparing with the same tools the students' answers and the characteristics of the tasks proposed in high school textbooks, they show that the difficulties identified in the students' answers exactly reflect the institutional limitations evidenced through the textbook analysis, what of course supports their theoretical position.

IV. Perspectives offered by the consideration of new educational contexts

The training of engineers is, from this point of view, interesting because the evolution of engineering practices induced by the technological evolution seriously questions the usual visions of mathematical learning and the ways these reflect in teaching practices. As stressed by P. Kent in (Artigue, Batanero, Kent, to appear), relying on the particular case of structural analysis in the training of civil engineers :

> « Structural analysis is essentially a deeply mathematical subject, and is central to the professional practice of a civil engineer, and thus also central to his or her education. As part of the design process, the engineer needs to predict the behaviour of a proposed structure (the walls of a building, a roof, a bridge, etc.). Before the advent of computers, the working life of an engineer, especially in the early part of his or her career, would be dominated by actually doing structural calculations using pen-and-paper, and a large part of the civil engineering degree was therefore dedicated to giving students an understanding and fluency in a variety of calculational techniques. For the majority of engineers today, all such calculations will be done in practice using computer software. The priority now for education is that students may "do" little mathematics themselves in the form of explicit calculation, yet they are likely to *use* more mathematics than ever before, implicit within computer software:
>
> "No longer do we have to grind through long calculations — the computer will do it for us. The challenge has changed from the ability to do this to the ability to interpret the meaning of mathematics to engineering and herein lies the challenge and change of emphasis." (Blockley & Woodman, 2002)
>
> A typical structural analysis curriculum of ten years ago would begin with a solid dose of analytical theory — using matrix algebra — followed later (perhaps in the second or third year) by working with computer software. Nowadays, work with computer software is likely to come much earlier, and the place of learning matrix algebra as a preparation for working with software is becoming open to question. »

According to him, what is asked today to the mathematicians in charge of such service courses, is to built learning strategies where the theoretical and technical mastery of the mathematical notions potentially involved is not considered as a necessary preliminary. More precisely:

> « What users of mathematics such as engineers are wanting however, and it seems that mathematicians will increasingly need to provide is a form of pull-based mathematics

> where the use of mathematical software makes mathematical ideas usable. It seems that carefully-designed use of IT can make it possible for students to use mathematical ideas before understanding techniques, and to make this part of a genuinely rounded mathematical experience (a few examples are given in Kent & Noss, 2003). »

There is there a radical and interesting change if one considers the role usually given to technological tools in mathematical learning. Technological tools indeed still tend to be seen only as pedagogical and didactical tools, not as consitutive elements of the learning process, contributing to the definition of its contents and forms. As if they had to be put at the service of some kind of universal mathematical contents and values. Such a common and resistant vision of technology seems insensitive to the fact that what we learn, beyond the way we learn it, is strongly dependent on our tool environment. Professionals, by the way they more and more reject the learning models that university mathematicians generally propose them, recall us this essential fact.

Another interesting evolution in the training of engineers is the increasing importance given to modelling, with a conception of modelling giving a crucial importance to the model elaboration, and not only to the mathematical work in the model as is generally the case in mathematics courses. Due to the technological evolution, what takes indeed an increasing importance in the competences engineers are asked to develop, is the ability to built, through appropriate languages, interfaces between the situations they face and the sophisticated technological tools they have at their disposal; and it is also the ability to analyse, interpret and discuss the data provided by the software they use. Such goals are more and more achieved through teaching strategies relying on professional projects where mathematics are involved together with a lot of other disciplines and have to find their adequate place, all the more as so many domains of knowledge are today part of the training of an engineer that the traditional organization of the training according to the disciplines becomes more and more difficult to implement, as also stressed by P. Kent:

> « There is broad agreement emerging amongst engineers that the way to deal with this "knowledge explosion" is to implement a shift in emphasis from teaching built around subject knowledge (i.e. the topics of engineering theory and science, and issues of professional practice) toward teaching about the process of engineering (how knowledge is operationalised), using engineering design as an organising and motivating principle of an engineering degree. »

What kind of mathematical learning can be generated by such didactic organizations close to professional practices and highly contextualized? What are their potential and limits? These questions are today widely open, in spite of the existence of some pioneering works as those by P. Kent and R. Noss mentioned above. They are not without interest for a more general reflection about mathematical learning at a time when, in many countries, curricula push towards the reinforcement of the links between mathematics and other disciplines, mathematics and society, through multidisciplinary projects and modelling activities [7]. These questions are also difficult. Mathematics living in other disciplines, in professional practices or in training practices close to these are mathematics difficult to identify and analyse (Bessot & Ridgway, 2000). They are most often intertwined with other forms of knowledge, supported by specific languages and representations, more and more encapsulated in sophisticated software that professionals use as black boxes. In order to be understood they require much more than the mathematical and didactical knowledge the didactic researchers usually rely on. Moreover they considerably vary from one domain to the other one. They thus require extensive and long term contact with new territories, as has been the case for C. Hoyles et R. Noss with banking (Noss & Hoyles, 1996), or A. Bessot & C. Laborde (Bessot & Laborde, 2005) with building industry. This can at least partially explain why, up to now, didactic research has not invested a lot these territories, even if such an investment can result very productive, by making us meet other forms of mathematical learning and question what we tend to take as evidences due to our limited field of experience.

V. Conclusion

This text only reflects very partially what is offered by research carried out at university level to our understanding of mathematical learning. I have chosen to focus on some current trends that I think especially promising. On the one hand, they renew usual visions of mathematical learning, of students' difficulties and of the didactic actions we could develop in order to improve the current situation. On the other hand, they show us that essential areas for mathematics education in our societies such as vocational education are still poorly addressed by didactic research, and push for their systematic investigation.

I will end this text by stressing my personal conviction that the issues raised by learning phenomena at university level are not so distant from those raised at more elementary levels of schooling, and that the research that these issues stimulate is also of interest for the wide majority of didacticians who try to understand and improve the learning and teaching of mathematics, from childhoold to adult learners.

Endnotes

[1]: PME : Psychology of Mathematics Education. PME is one of the affiliated study groups of ICMI, the International Commission for Mathematical Instruction.

[2]: PME-NA is the North American branch of PME.

[3]: The first Handbook was coordinated by D.A. Grouws (Grouws, 1992).

[4]: Other directions are explored in (Artigue, Batanero, Kent, to appear) as for instance the renewal offered by an embodied vision of cognition or by the exponential increase of research on the teaching and learning of statistics.

[5]: A specific reflection about the notion of setting, its use by didacticians, and its relationships with the notion of semiotic register is presented in the proceedings of the conference organized on this specific theme in Paris in 2000 for honouring R. Douady (Perrin-Glorian & Robert, 2001).

[6]: In the Chevallard's anthropological approach (Chevallard, 1992, 1999), mathematical practices are described in terms of praxeologies. A praxeology is a four-uplet (T,τ,θ,Θ), T labelling a type of task, τ a technique that is to say a way for realizing T, θ a technology that is to say a discourse for describing, explaining and justifying the technique, and Θ a theory that plays the role of technology for the technology. A mathematical organization is said pointwise if it is generated by only one type of task (and can be analysed through one praxeology), local if it results from the integration of several pointwise organizations around a common technological discourse, regional if it results from the integration of several local organizations around a common theory. Starting from these notions, the authors define what they call a local mathematical organization nearly complete, by imposing to its institutional construction the satisfaction of six different criteria. These determine what has to be offered by this organization for what Chevallard has identified as the necessary moments of a study process in mathematics: first meeting, exploration, working out of the technique, technological questioning, institutionalization, evaluation.

[7]: This phenomenon is at the source of the on-going ICMI Study 14 titled: *Applications and Modelling in Mathematics Education* (cf. (Henn & Blum, 2004) for the pre-proceedings of the associated Conference).

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